

Research article

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On some classes of generalized Schrödinger equations

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Abstract: Some classes of generalized Schrödinger stationary problems are studied. Under appropriated conditions is proved the existence of at least $1 + \sum_{i=2}^m \dim V_{\lambda_i}$ pairs of nontrivial solutions if a parameter involved in the equation is large enough, where V_{λ_i} denotes the eigenspace associated to the i -th eigenvalue λ_i of laplacian operator with homogeneous Dirichlet boundary condition.

Keywords: Generalized Schrödinger problems, multiplicity of solutions, Nehari manifold

MSC: 35J10, 35J25, 35J60.

1 Introduction

Recently, it has been studied in [19] the following class of generalized Schrödinger problems

$$\begin{cases} -\operatorname{div}(\vartheta(u)\nabla u) + \frac{1}{2}\vartheta'(u)|\nabla u|^2 = \lambda|u|^{q-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (P_{\lambda,q})$$

where $\Omega \subset \mathbf{R}^N$, $N \geq 3$, is a bounded smooth domain, q, λ are real parameters and $\vartheta : \mathbf{R} \rightarrow [1, \infty)$ is an even C^1 -function verifying:

- (ϑ_1) $t \mapsto \vartheta(t)$ is decreasing in $(-\infty, 0)$ and increasing in $(0, \infty)$;
- (ϑ_2) $t \mapsto \vartheta(t)/t^2$ increasing in $(-\infty, 0)$ and decreasing in $(0, \infty)$;
- (ϑ_3) $\lim_{|t| \rightarrow \infty} \vartheta(t)/t^2 = \alpha^2/2$, for some $\alpha > 0$.

By considering the ordinary differential equation

$$f'(s) = \frac{1}{\vartheta(f(s))^{1/2}} \text{ and } f(0) = 0, \quad (\text{ODE})$$

whose unique solution is $f(s) = Y^{-1}(s)$, with $Y(t) := \int_0^t \vartheta(r)^{1/2} dr$, the authors proved that

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Proposition 1.1. *The following claims hold:*

- (i) f is an increasing C^2 -diffeomorphism, with $f''(s) = -\vartheta'(f(s))/2\vartheta(f(s))^2$;
- (ii) $0 < f'(s) \leq 1$, for all $s \in \mathbf{R}$;
- (iii) $\lim_{s \rightarrow 0} f(s)/s = 1/\vartheta(0)^{1/2}$;
- (iv) $|f(s)| \leq |s|$, for all $s \in \mathbf{R}$;
- (v) Suppose that $(\vartheta_1) - (\vartheta_2)$ hold. Then, $|f(s)|/2 \leq f'(s)|s| \leq |f(s)|$, for all $s \in \mathbf{R}$, and the map $s \mapsto |f(s)|/\sqrt{|s|}$ is nonincreasing in $(-\infty, 0)$ and nondecreasing in $(0, \infty)$;
- (vi) Suppose that $(\vartheta_1) - (\vartheta_3)$ hold. Then,

$$\lim_{|s| \rightarrow \infty} \frac{|f(s)|}{\sqrt{|s|}} = \left(\frac{8}{\alpha^2}\right)^{1/4} \quad \text{and} \quad \lim_{|s| \rightarrow \infty} \frac{f(s)}{s} = 0,$$

where α is given in (ϑ_3) .

Motivated by ideas in [5] and [14], the authors in [19] make use of the following approach: Despite the energy functional associated to $(P_{\lambda,q})$ is not well defined in $H_0^1(\Omega)$, Proposition 1.1 allows them to consider the change of variable $v = f(u)$ in the semilinear problem $(P_{\lambda,q})$ in order to obtain the problem

$$\begin{cases} -\Delta v = \lambda g(v) & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \quad (P'_{\lambda,q})$$

where $g(s) := f'(s)|f(s)|^{q-2}f(s)$, which has the advantage of possessing a well defined C^1 -energy functional in $H_0^1(\Omega)$, given by

$$I_{\lambda,q}(v) = \frac{1}{2}\|v\|^2 - \lambda \int_{\Omega} G(v)dx, \quad (1.1)$$

where $\|v\|^2 := \int_{\Omega} |\nabla v|^2 dx$ and $G(s) = \int_0^s g(r)dx = (1/q)|f(s)|^q$. They also prove that critical points of (1.1) in $C^1(\bar{\Omega})$ are weak solutions of $(P_{\lambda,q})$. In this way, by working with $(P'_{\lambda,q})$, among other things, they were able to prove that:

- (a) If $q = 2$, $\lambda > \vartheta(0)\lambda_1$ and $(\vartheta_1) - (\vartheta_2)$ holds, then $(P_{\lambda,q})$ has a unique positive solution;
- (b) If $q = 4$, $\lambda > (\alpha^2/4)\lambda_1$ and $(\vartheta_1) - (\vartheta_3)$ holds, then $(P_{\lambda,q})$ has at least one positive solution.

Having in mind the previous results, the present paper has as its main goal to improve the results in [19] when one considers the cases $q = 2$ or $q = 4$ in problem $(P_{\lambda,q})$. Indeed, since in these two particular cases we prove in Lemma 2.1 that g is asymptotically linear at zero and at infinity, respectively, by using genus theory combined with arguments involving the Nehari manifold, it is possible to show that the number of solutions increases with λ . To be more precise, if $\dim V_{\lambda_i}$ denotes the dimension of the eigenspace V_{λ_i} associated to i -th eigenvalue λ_i of laplacian operator under homogeneous Dirichlet boundary condition, we prove the following multiplicity result:

Theorem 1.2. *Suppose that $(\vartheta_1) - (\vartheta_3)$ hold.*

- (i) If $q = 2$ and $\lambda > \vartheta(0)\lambda_m$, then problem $(P_{\lambda,q})$ possesses at least $1 + \sum_{i=2}^m \dim V_{\lambda_i}$ pairs of nontrivial solutions u_i with $I_{\lambda,2}(f^{-1}(u_i)) > 0$;
- (ii) If $q = 4$ and $\lambda > (\alpha^2/4)\lambda_m$, then problem $(P_{\lambda,q})$ has at least $1 + \sum_{i=2}^m \dim V_{\lambda_i}$ pairs of nontrivial solutions u_i with $I_{\lambda,4}(f^{-1}(u_i)) > 0$.

By comparing Theorem 1.2 in [19] with Theorem 1.2(i) previously aimed, we can immediately conclude that at least $\sum_{i=2}^m \dim V_{\lambda_i}$ of solutions provided in Theorem 1.2(i) are sign-changing.

To better understand the relevance of Schrödinger equations in different fields of applied science, we refer to [1–3, 10–13, 15, 17]. For a brief history about stationary Schrödinger equations (generalized or not), see [5–9, 14, 16, 20–22, 25].

The paper is organized in a unique section where we study both cases, $q = 2$ and $q = 4$.

2 Multiplicity of solutions

Since, by [19], it does not exist any nontrivial solution when $\lambda \leq 0$, along of this section we are just considering positive values of λ . Moreover, before proving the main results of this section we need to study the properties of function g . Such properties play an important role throughout the paper.

Lemma 2.1. *Suppose that $(\vartheta_1) - (\vartheta_3)$ hold. Then:*

- (i) *Map $s \mapsto |g(s)|$ is decreasing in $(-\infty, 0)$, increasing in $(0, \infty)$, $\lim_{s \rightarrow 0} g(s)/s = 1/\vartheta(0)$ and $\lim_{|s| \rightarrow \infty} |g(s)| = \sqrt{2}/\alpha$, if $q = 2$;*
- (ii) *Map $s \mapsto g(s)/s$ is decreasing in $(-\infty, 0)$, increasing in $(0, \infty)$, $\lim_{s \rightarrow 0} g(s)/s = 0$ and $\lim_{|s| \rightarrow \infty} g(s)/s = 4/\alpha^2$, if $q = 4$;*
- (iii) *Map $s \mapsto (1/2)g(s)s - G(s)$ is decreasing in $(-\infty, 0)$, increasing in $(0, \infty)$ and $\lim_{|s| \rightarrow \infty} [(1/2)g(s)s - G(s)] = +\infty$, if $q = 4$.*

Proof. (i) The monotonicity is a straightforward consequence of Proposition 1.1(ii) and (ϑ_2) . On the other hand, by Proposition 1.1(iii)

$$\lim_{s \rightarrow 0} \frac{g(s)}{s} = \lim_{s \rightarrow 0} \frac{1}{\vartheta(f(s))^{1/2}} \times \frac{f(s)}{s} = \frac{1}{\vartheta(0)}.$$

Moreover, by (ϑ_3)

$$\lim_{|s| \rightarrow \infty} |g(s)| = \lim_{|s| \rightarrow \infty} \frac{1}{(\vartheta(f(s))/f(s)^2)^{1/2}} = \frac{\sqrt{2}}{\alpha}.$$

(ii) Since f is odd (because ϑ is even), it is sufficient to prove this item for $s > 0$. Observe that

$$\frac{g(s)}{s} = \frac{f(s)^3}{s\vartheta(f(s))^{1/2}} = \frac{t^2}{Y(t)} \times \frac{t}{\vartheta(t)^{1/2}},$$

where $t := f(s)$ and $Y(t) := \int_0^t \vartheta(r)^{1/2} dr$. It follows from (ϑ_2) that $t/\vartheta(t)^{1/2}$ (and consequently $t^2/Y(t)$) is increasing in $(0, \infty)$. This proves that $g(s)/s$ is increasing in $(0, \infty)$. Moreover, by item (iii) and (vi) of Proposition 1.1, we have

$$\lim_{s \rightarrow 0} \frac{g(s)}{s} = \lim_{s \rightarrow 0} \frac{f(s)^2}{\vartheta(f(s))^{1/2}} \times \frac{f(s)}{s} = 0$$

and

$$\lim_{s \rightarrow \infty} \frac{g(s)}{s} = \lim_{s \rightarrow \infty} \left(\frac{f(s)}{s^{1/2}}\right)^2 \times \frac{f(s)}{\vartheta(f(s))^{1/2}} = \left(\frac{8}{\alpha^2}\right)^{1/2} \times \frac{\sqrt{2}}{\alpha} = \frac{4}{\alpha^2}.$$

(iii) The monotonicity follows immediately from (ii). To prove the second part, note that

$$\frac{1}{2}g(s)s - G(s) = \frac{t^3}{4\vartheta(t)^{1/2}} \left(2Y(t) - t\vartheta(t)^{1/2}\right).$$

By (ϑ_3) we know that $t^3/4\vartheta(t)^{1/2}$ goes to infinity as t goes to infinity. On the other hand, by (ϑ_2) , $2Y(t) - t\vartheta(t)^{1/2}$ is nonnegative and increasing in $(0, \infty)$. Indeed, by defining $h(t) := 2Y(t) - t\vartheta(t)^{1/2}$, we have $h(0) = 0$ and

$$h'(t) = \frac{2\vartheta(t) - t\vartheta'(t)}{2\vartheta(t)^{1/2}} > 0, \forall t > 0.$$

The result follows. □

From now on $\{e_j\}$ stands for a Hilbertian basis of $H_0^1(\Omega)$ composed by eigenfunctions of the laplacian operator with homogeneous Dirichlet boundary condition, V_{λ_j} is the eigenspace associated to λ_j , \mathbb{S} and $\mathbb{S}_{d(m)}$ are, respectively, the unit sphere of $H_0^1(\Omega)$ and the unit sphere of $W_m := \bigoplus_{j=1}^m V_{\lambda_j}$.

2.1 Case $q = 2$

Proof of Theorem 1.2(i):

By Lemma 2.1(i) and Sobolev embedding

$$I_{\lambda,2}(v) \geq \frac{1}{2} \|v\|^2 - \frac{\sqrt{2}\lambda}{\alpha} \int_{\Omega} |v| dx \geq \frac{1}{2} \|v\|^2 - \frac{\sqrt{2}C\lambda}{\alpha} \|v\|.$$

Therefore $I_{\lambda,2}$ is coercive. Since $I_{\lambda,2}$ is weakly lower semicontinuous, we conclude that $I_{\lambda,2}$ is bounded from below. On the other hand, since

$$\frac{I_{\lambda,2}(sv)}{s^2} = \frac{1}{2} - \lambda \int_{[v \neq 0]} \left[\frac{G(sv)}{(sv)^2} \right] v^2 dx,$$

for all $v \in d(m)$. We conclude from Lemma 2.1(i), L'Hospital and Lebesgue Dominated Convergence Theorem that

$$\lim_{s \rightarrow 0} \frac{I_{\lambda,2}(sv)}{s^2} = \frac{1}{2} - \frac{\lambda}{2g(0)} \int_{\Omega} v^2 dx,$$

for all $v \in d(m)$. Since $v = \sum_{j=1}^{d(m)} v_{\lambda_j} e_j$, where $d(m) := 1 + \sum_{i=2}^m \dim V_{\lambda_i}$, we get

$$\lim_{s \rightarrow 0} \frac{I_{\lambda,2}(sv)}{s^2} = \frac{1}{2} - \frac{\lambda}{2g(0)} \left(\sum_{j=1}^{d(m)} v_j^2 \int_{\Omega} e_j^2 dx + \sum_{j \neq i} v_j v_i \int_{\Omega} e_j e_i dx \right) = \frac{1}{2} - \frac{\lambda}{2g(0)} \sum_{j=1}^{d(m)} \left(\frac{v_j^2}{\lambda_j} \right).$$

Since $v \in \mathcal{S}_{d(m)}$,

$$\lim_{s \rightarrow 0} \frac{I_{\lambda,2}(sv)}{s^2} = \frac{1}{2} \sum_{j=1}^{d(m)} \left[1 - \frac{\lambda}{g(0)\lambda_j} \right] v_j^2 \leq \frac{1}{2} \left(1 - \frac{\lambda}{g(0)\lambda_m} \right) < 0,$$

for all $v \in \mathcal{S}_{d(m)}$, because $\lambda > g(0)\lambda_m$. Therefore, there exist $\varepsilon, \delta > 0$ such that

$$I_{\lambda,2}(sv) = (I_{\lambda,2}(sv)/s^2)s^2 \leq -\varepsilon s^2,$$

for all $0 < s < \delta$ and $v \in \mathcal{S}_{d(m)}$. Fixing $0 < s_* < \delta$, we have

$$\sup_{w \in s_* \mathcal{S}_{d(m)}} I_{\lambda,2}(w) < 0.$$

Since $I_{\lambda,2}$ is coercive, it is standard to prove that it satisfies the $(PS)_c$ condition. Finally, as $I_{\lambda,2}$ is an even C^1 -functional, it follows from Theorem 9.1 in [18] (see also [4]) that $I_{\lambda,2}$ has at least $d(m)$ pairs of critical points. \square

2.2 Case $q = 4$

Before we are ready to prove Theorem 1.2(ii), we will make a careful study about some topological and geometrical aspects involving the Nehari Manifold. Let

$$\mathcal{N} = \left\{ v \in H_0^1(\Omega) \setminus \{0\} : \|v\|^2 = \lambda \int_{\Omega} g(v)v dx \right\}$$

be the Nehari manifold associated to $I_{\lambda,4}$, \mathcal{S} the unit sphere in $H_0^1(\Omega)$ and

$$\mathcal{F} := \left\{ v \in H_0^1(\Omega) : \|v\|^2 < \frac{4\lambda}{\alpha^2} \int_{\Omega} v^2 dx \right\}.$$

Lemma 2.2. *If ϑ satisfies $(\vartheta_1) - (\vartheta_3)$ and $\lambda > (\alpha^2/4)\lambda_1$, the following claims hold:*

- (i) *The set \mathcal{F} is open and nonempty;*
- (ii) $\partial\mathcal{F} = \{v \in H_0^1(\Omega) : \|v\|^2 = (4\lambda/\alpha^2) \int_{\Omega} v^2 dx\}$;
- (iii) $\mathcal{F}^c = \{v \in H_0^1(\Omega) : \|v\|^2 \geq (4\lambda/\alpha^2) \int_{\Omega} v^2 dx\}$;
- (iv) $\mathcal{N} \subset \mathcal{F}$;
- (v) $\mathcal{S} \cap \mathcal{F} \neq \emptyset$.

Proof. (i) Since $\lambda > (\alpha^2/4)\lambda_1$, any eigenfunction associated to λ_1 belongs to \mathcal{F} . Moreover, $\mathcal{F} = \Phi^{-1}(-\infty, 0)$ where $\Phi : H_0^1(\Omega) \rightarrow \mathbb{R}$ is the continuous functional defined by $\Phi(v) = \|v\|^2 - (4\lambda/\alpha^2) \int_{\Omega} v^2 dx$. Items (ii) and (iii) are immediate.

(iv) If $v \in \mathcal{N}$ then, by Lemma 2.1(ii), we obtain

$$\|v\|^2 = \lambda \int_{[v \neq 0]} \left[\frac{g(v)}{v} \right] v^2 dx < \frac{4\lambda}{\alpha^2} \int_{\Omega} v^2 dx.$$

(v) It is sufficient to choose a normalized (in $H_0^1(\Omega)$) eigenfunction associated to λ_1 . □

By previous Lemma, the set $\mathcal{S}_{\mathcal{F}} := \mathcal{S} \cap \mathcal{F}$ is open in \mathcal{S} . Moreover, $\partial\mathcal{S}_{\mathcal{F}} = \{v \in \mathcal{S} : 1 = (4\lambda/\alpha^2) \int_{\Omega} v^2 dx\}$ and $\mathcal{S}_{\mathcal{F}}^c = \{v \in \mathcal{S} : 1 \geq (4\lambda/\alpha^2) \int_{\Omega} v^2 dx\}$ are nonempty because any normalized eigenfunction associated to λ_j such that $\lambda \leq (\alpha^2/4)\lambda_j$, belongs to $\mathcal{S}_{\mathcal{F}}^c$. Thus, $\mathcal{S}_{\mathcal{F}}$ is a noncomplete C^1 -submanifold of $H_0^1(\Omega)$.

Lemma 2.3. *Suppose that ϑ verifies $(\vartheta_1) - (\vartheta_3)$ and let $h_v : [0, \infty) \rightarrow \mathbb{R}$ be defined by $h_v(s) = I_{\lambda,4}(sv)$.*

- (i) *For each $v \in \mathcal{F}$, there exists a unique $s_v > 0$ such that $h'_v(s) > 0$ in $(0, s_v)$, $h'_v(s_v) = 0$ and $h'_v(s) < 0$ in (s_v, ∞) . Moreover, $sv \in \mathcal{N}$ if, and only if, $s = s_v$;*
- (ii) *For each $v \in \mathcal{F}^c \setminus \{0\}$, $h'_v(s) > 0$ for all $s \in (0, \infty)$.*

Proof. (i) Observe that $h_v(0) = 0$. Moreover, for each $v \in \mathcal{F}$, we have

$$\frac{h_v(s)}{s^2} = \frac{1}{2} \|v\|^2 - \lambda \int_{[v \neq 0]} \left[\frac{G(sv)}{(sv)^2} \right] v^2 dx. \tag{2.1}$$

Thus, in view of Lemma 2.1(ii), L'Hôpital rule and Lebesgue's dominated convergence theorem, it follows that

$$\lim_{s \rightarrow \infty} \frac{h_v(s)}{s^2} = \frac{1}{2} \left[\|v\|^2 - (4\lambda/\alpha^2) \int_{\Omega} v^2 dx \right] < 0.$$

Showing that $\lim_{s \rightarrow \infty} h_v(s) = -\infty$. Moreover, $h_v(s)$ is positive for s small enough. Indeed, reasoning as in the previous limit, we get

$$\lim_{s \rightarrow 0^+} \frac{h_v(s)}{s^2} = \frac{1}{2} \|v\|^2 - \lambda \int_{[v \neq 0]} \left[\frac{G(sv)}{(sv)^2} \right] v^2 dx = \frac{1}{2} \|v\|^2 > 0.$$

Hence, there exists a global maximum point $s_v > 0$ of h_v which, by Lemma 2.1(ii), is the unique critical point of h_v .

(ii) If $v \in \mathcal{F}^c \setminus \{0\}$, then $\|v\|^2 \geq (4\lambda/\alpha^2) \int_{\Omega} v^2 dx$. Thus, by Lemma 2.1(ii), it follows that

$$\frac{h'_v(s)}{s} = \|v\|^2 - \lambda \int_{[v \neq 0]} \frac{g(sv)}{sv} v^2 dx \geq \lambda \int_{[v \neq 0]} \left[\frac{4}{\alpha^2} - \frac{g(sv)}{sv} \right] v^2 dx > 0, \quad \forall s > 0.$$

Consequently, $h'_v(s) > 0$ for all $s \in (0, \infty)$. □

Lemma 2.4. *If \mathcal{G} verifies $(\mathcal{G}_1) - (\mathcal{G}_3)$, the following claims hold:*

- (A₁) *There exists $\tau > 0$ such that $s_v \geq \tau$, for all $v \in \mathcal{S}_{\mathcal{F}}$;*
- (A₂) *For each compact set $\mathcal{W} \subset \mathcal{S}_{\mathcal{F}}$ there exists $C_{\mathcal{W}} > 0$ such that $s_v \leq C_{\mathcal{W}}$, for all $v \in \mathcal{W}$;*
- (A₃) *The map $\widehat{m} : \mathcal{F} \rightarrow \mathcal{N}$ given by $\widehat{m}(v) = s_v v$ is continuous and $m := \widehat{m}|_{\mathcal{S}_{\mathcal{F}}}$ is a homeomorphism between $\mathcal{S}_{\mathcal{F}}$ and \mathcal{N} . Moreover, $m^{-1}(v) = v/\|v\|$.*

Proof. (A₁) Suppose that there exists $\{v_n\} \subset \mathcal{S}_{\mathcal{F}}$ with $s_n := s_{v_n} \rightarrow 0$. In this case, we get $v \in H_0^1(\Omega)$ with $v_n \rightarrow v$ in $H_0^1(\Omega)$. It follows from Lemma 2.1(ii) that

$$1 = \lambda \int_{\Omega} g(s_n v_n) v_n dx \leq (4/\alpha^2) \lambda s_n \int_{\Omega} v_n^2 dx. \tag{2.2}$$

By passing to the limit as $n \rightarrow \infty$ in the last inequality, we get a contradiction.

(A₂) Let $\{v_n\} \subset \mathcal{W}$ be a sequence such that $s_n := s_{v_n} \rightarrow \infty$. Since \mathcal{W} is compact, up to a subsequence, we get $v \in \mathcal{W}$ such that $v_n \rightarrow v$ in $H_0^1(\Omega)$. Hence, passing to the lower limit as $n \rightarrow \infty$ in

$$1 = \|v_n\|^2 \geq \lambda \int_{[v \neq 0]} \frac{g(s_n v_n)}{s_n v_n} v_n^2 \chi_{[v_n \neq 0]} dx,$$

it follows from Lemma 2.1(ii) that

$$\|v\|^2 = 1 \geq (4\lambda/\alpha^2) \int_{\Omega} v^2 dx,$$

showing that $v \in \mathcal{F}^c$. Since $v \in \mathcal{W} \subset \mathcal{F}$, we have a contradiction.

(A₃) We are going to prove that \widehat{m} is continuous. Let $\{v_n\} \subset \mathcal{F}$ and $v \in \mathcal{F}$ be such that $v_n \rightarrow v$ in $H_0^1(\Omega)$. Since $\widehat{m}(sv) = \widehat{m}(w)$ for all $w \in \mathcal{F}$ and $s > 0$, we can assume that $\{v_n\} \subset \mathcal{S}_{\mathcal{F}}$. Hence,

$$s_n = s_n \|v_n\|^2 = \lambda \int_{\Omega} g(s_n v_n) v_n dx, \tag{2.3}$$

where $s_n := s_{v_n}$. By (A₁) and (A₂), it follows that, passing to a subsequence, $s_n \rightarrow s > 0$. Thence, passing to the limit as $n \rightarrow \infty$ in (2.3), we have

$$s = s \|v\|^2 = \lambda \int_{\Omega} g(sv) v dx,$$

showing that $\widehat{m}(v_n) = s_n v_n \rightarrow sv = \widehat{m}(v)$. The second part of (A₃) is immediate. □

Lemma 2.5. *Suppose that \mathcal{G} satisfies $(\mathcal{G}_1) - (\mathcal{G}_3)$. Then $I_{\lambda,4}$ is bounded from below in \mathcal{N} .*

Proof. By Lemma 2.1(iii), we get

$$I_{\lambda,4}(v) = \lambda \int_{\Omega} \left[\frac{1}{2} g(v)v - G(v) \right] dx \geq 0, \quad \forall v \in \mathcal{N}. \tag{2.4}$$

Therefore $I_{\lambda,4}$ is bounded from below in \mathcal{N} . □

Now we are going to set the maps $\widehat{\Psi}_{\lambda,4} : \mathcal{F} \rightarrow \mathbb{R}$ and $\Psi_{\lambda,4} : \mathcal{S}_{\mathcal{F}} \rightarrow \mathbb{R}$, by

$$\widehat{\Psi}_{\lambda,4}(u) = I_{\lambda,4}(\widehat{m}(u)) \text{ and } \Psi_{\lambda,4} := (\widehat{\Psi}_{\lambda,4})|_{\mathcal{S}_{\mathcal{F}}}.$$

Previous functions have important properties which will be stated in the next lemma. The proof is a direct consequence of Lemmas 2.3 and 2.4, see [24].

Lemma 2.6. *Suppose that ϑ verifies $(\vartheta_1) - (\vartheta_3)$. Then,*

(i) $\widehat{\Psi}_{\lambda,4} \in C^1(\mathcal{F}, \mathbb{R})$ and

$$\widehat{\Psi}'_{\lambda,4}(u)v = \frac{\|\widehat{m}(u)\|}{\|u\|} I'_{\lambda,4}(\widehat{m}(u))v, \quad \forall u \in \mathcal{F} \text{ and } \forall v \in H_0^1(\Omega).$$

(ii) $\Psi_{\lambda,4} \in C^1(\mathcal{S}_{\mathcal{F}}, \mathbb{R})$ and

$$\Psi'_{\lambda,4}(u)v = \|m(u)\| I'_{\lambda,4}(m(u))v, \quad \forall v \in T_u \mathcal{S}_{\mathcal{F}}.$$

(iii) *If $\{u_n\}$ is a $(PS)_c$ sequence for $\Psi_{\lambda,4}$ then $\{m(u_n)\}$ is a $(PS)_c$ sequence for $I_{\lambda,4}$. If $\{u_n\} \subset \mathcal{N}$ is a bounded $(PS)_c$ sequence for $I_{\lambda,4}$ then $\{m^{-1}(u_n)\}$ is a $(PS)_c$ sequence for $\Psi_{\lambda,4}$.*

(iv) *u is a critical point of $\Psi_{\lambda,4}$ if, and only if, $m(u)$ is a nontrivial critical point of $I_{\lambda,4}$. Moreover, the corresponding critical values coincide and*

$$\inf_{\mathcal{S}_{\mathcal{F}}} \Psi_{\lambda,4} = \inf_{\mathcal{N}} I_{\lambda,4}.$$

Proposition 2.7. *Suppose that $(\vartheta_1) - (\vartheta_3)$ hold. If $\{v_n\} \subset \mathcal{S}_{\mathcal{F}}$ is such that $\text{dist}(v_n, \partial \mathcal{S}_{\mathcal{F}}) \rightarrow 0$, then there exists $v \in H_0^1(\Omega) \setminus \{0\}$ such that $v_n \rightarrow v$ in $H_0^1(\Omega)$, $s_{v_n} \rightarrow \infty$ and*

$$\Psi_{\lambda,4}(v_n) \rightarrow \infty. \tag{2.5}$$

Proof. Since $\{v_n\} \subset \mathcal{S}_{\mathcal{F}}$ is bounded, up to a subsequence, there exists $v \in H_0^1(\Omega)$ with $v_n \rightarrow v$ in $H_0^1(\Omega)$. Since $\text{dist}(v_n, \partial \mathcal{S}_{\mathcal{F}}) \rightarrow 0$, there exists $\{w_n\} \subset \partial \mathcal{S}_{\mathcal{F}}$ such that $\|v_n - w_n\| \rightarrow 0$ as $n \rightarrow \infty$. Thus,

$$\begin{aligned} \left| (4\lambda/\alpha^2) \int_{\Omega} v_n^2 dx - 1 \right| &= \left| (4\lambda/\alpha^2) \int_{\Omega} (v_n^2 - w_n^2) dx \right| \\ &\leq (4\lambda/\alpha^2) \|v_n + w_n\|_2 \|v_n - w_n\|_2 \\ &\leq (8\lambda/\alpha^2 \lambda_1) \|v_n - w_n\|. \end{aligned}$$

Therefore,

$$(4\lambda/\alpha^2) \int_{\Omega} v_n^2 dx \rightarrow 1.$$

By using compact embedding from $H_0^1(\Omega)$ into $L^2(\Omega)$, it follows that

$$1 = (4\lambda/\alpha^2) \int_{\Omega} v^2 dx. \tag{2.6}$$

Thus $v \neq 0$. Suppose by contradiction that, for some subsequence, $\{s_{v_n}\}$ is bounded. In this case, passing again to a subsequence, there exists $s_0 > 0$ (see Lemma 2.4(A₁)) such that

$$s_{v_n} \rightarrow s_0. \tag{2.7}$$

It follows from (2.7) and

$$s_{v_n} = \lambda \int_{\Omega} g(s_{v_n} v_n) v_n dx,$$

that

$$s_0 = \lambda \int_{\Omega} g(s_0 v) v dx.$$

Combining last equality and Lemma 2.1(ii), we obtain

$$1 < (4\lambda/\alpha^2) \int_{\Omega} v^2 dx.$$

But this contradicts (2.6). Showing that $s_{v_n} \rightarrow \infty$. Finally, from $s_{v_n} \rightarrow \infty$, Lemma 2.1(iii) and Fatou Lemma, we get

$$\liminf_{n \rightarrow \infty} \Psi_{\lambda,4}(v_n) = \lambda \liminf_{n \rightarrow \infty} \int_{\Omega} \left[\frac{1}{2} g(s_{v_n} v_n) s_{v_n} v_n - G(s_{v_n} v_n) \right] dx \geq \infty.$$

□

Proposition 2.8. *Suppose that $(\vartheta_1) - (\vartheta_3)$ hold and $\lambda > (\alpha^2/4)\lambda_1$. Then $\Psi_{\lambda,4}$ satisfies the $(PS)_c$ condition.*

Proof. By Lemmas 2.4(A₃) and 2.6(iii), it is sufficient to show that $I_{\lambda,4}$ satisfies the $(PS)_c$ condition. For this, let $\{w_n\} \subset \mathcal{N}$ be a $(PS)_c$ sequence for $I_{\lambda,4}$. We are going to prove that $\{w_n\}$ is bounded in $H_0^1(\Omega)$. Indeed, otherwise, up to a subsequence, we have $\|w_n\| \rightarrow \infty$. Define $v_n := w_n/\|w_n\| = m^{-1}(w_n) \in \mathcal{S}_{\mathcal{F}}$. Thus $\{v_n\}$ is bounded in $H_0^1(\Omega)$ and

$$\Psi_{\lambda,4}(v_n) \rightarrow c. \tag{2.8}$$

Consequently, there exists $v \in H_0^1(\Omega)$ such that

$$v_n \rightharpoonup v \text{ in } H_0^1(\Omega). \tag{2.9}$$

Suppose by contradiction that $v = 0$. Since $\{\Psi_{\lambda,4}(v_n)\}$ is bounded, it follows that there exists $C > 0$ such that

$$C > \Psi_{\lambda,4}(v_n) = I_{\lambda,4}(s_{v_n} v_n) \geq I_{\lambda,4}(s v_n) = \left[\frac{1}{2} - \lambda \int_{[v_n \neq 0]} \frac{G(s v_n)}{(s v_n)^2} v_n^2 dx \right] s^2, \forall s > 0. \tag{2.10}$$

By Lemma 2.1(ii), L'Hôpital rule and compact embedding, passing to the limit as $n \rightarrow \infty$ in (2.10), we get

$$C \geq (1/2)s^2, \forall s > 0,$$

a clear contradiction. Thereby, we conclude that $v \neq 0$.

Since $\{w_n\} \subset \mathcal{N}$ is a $(PS)_c$ sequence for functional $I_{\lambda,4}$, we get

$$o_n(1) + \int_{\Omega} \nabla w_n \nabla w dx = \lambda \int_{\Omega} g(w_n) w dx, \forall w \in H_0^1(\Omega).$$

Dividing last equality by $\|w_n\|$, we have

$$o_n(1) + \int_{\Omega} \nabla v_n \nabla w dx = \lambda \int_{[v_n \neq 0]} \left[\frac{g(\|w_n\| v_n)}{\|w_n\| v_n} \right] v_n w dx.$$

Passing to the limit as $n \rightarrow \infty$, it follows from Lemma 2.1(ii) that

$$\int_{\Omega} \nabla v \nabla w dx = (4\lambda/\alpha^2) \int_{\Omega} v w dx, \forall w \in H_0^1(\Omega). \tag{2.11}$$

Now we are going to consider two cases:

(i) If $(4\lambda/\alpha^2) \neq \lambda_j$, whatever $j > 1$, it follows from (2.11) that $v = 0$. But this is a contradiction. Therefore $\{w_n\}$ is bounded in $H_0^1(\Omega)$.

(ii) If $(4\lambda/\alpha^2) = \lambda_j$, for some $j > 1$, then (2.11) implies that v is an eigenfunction associated to λ_j . From (2.11), it follows also that $\|v\|^2 = (4\lambda/\alpha^2) \int_{\Omega} v^2 dx$, i.e., $v \in \partial\mathcal{F}$. On the other hand,

$$(4\lambda/\alpha^2) \int_{\Omega} v^2 dx = \|v\|^2 \leq \liminf_{n \rightarrow \infty} \|v_n\|^2 = 1.$$

Suppose that

$$\|v\|^2 = (4\lambda/\alpha^2) \int_{\Omega} v^2 dx < 1. \tag{2.12}$$

In this case, since

$$\|w_n\| = \|s_{v_n} v_n\| = s_{v_n}, \tag{2.13}$$

passing to the limit as $n \rightarrow \infty$ in

$$\Psi_{\lambda,4}(v_n) = \|w_n\|^2 \left\{ \frac{1}{2} - \lambda \int_{\Omega} \left[\frac{G(\|w_n\|v_n)}{(\|w_n\|v_n)^2} \right] v_n^2 dx \right\}$$

and using Lemma 2.1(ii), L'Hôpital rule and (2.12), we conclude that $\Psi_{\lambda,4}(v_n) \rightarrow \infty$, a contradiction with (2.8). Consequently,

$$\|v\|^2 = (4\lambda/\alpha^2) \int_{\Omega} v^2 dx = 1, \tag{2.14}$$

showing that $v = e_j$ and

$$\|v_n\| \rightarrow \|v\|. \tag{2.15}$$

By using (2.9) and (2.15), we derive $v_n \rightarrow v$ in $H_0^1(\Omega)$ with $v \in \partial\mathcal{S}_{\mathcal{F}}$ (see (2.14)). Invoking Proposition 2.7, we conclude that

$$\Psi_{\lambda,4}(v_n) \rightarrow \infty. \tag{2.16}$$

Since (2.16) cannot occurs, we conclude that $\{v_n\}$ is bounded.

Hence, there exists $v \in H_0^1(\Omega)$ such that $v_n \rightarrow v$ in $H_0^1(\Omega)$ up to a subsequence. Since $v_n \rightarrow v$, to finish the proof we just have to prove that $\|v_n\| \rightarrow \|v\|$. To this end, it is sufficient to note that since $\{v_n\}$ is a $(PS)_c$ sequence, we have

$$o_n(1) + \int_{\Omega} \nabla v_n \nabla v dx = \lambda \int_{\Omega} g(v_n) v dx.$$

Passing to the limit as $n \rightarrow \infty$ in the previous equality, we get

$$\|v\|^2 = \lambda \int_{\Omega} g(v) v dx. \tag{2.17}$$

Then (2.17) and Lebesgue's convergence theorem imply that

$$\|v_n\|^2 = \lambda \int_{\Omega} g(v_n) v_n dx = \lambda \int_{\Omega} g(v) v dx + o_n(1) = \|v\|^2 + o_n(1).$$

□

The main result of this section will be proved through Krasnoselski's genus theory. For this, we start defining some preliminaries notations:

$$y_j := \{B \in \mathcal{E} : B \subset \mathcal{S}_{\mathcal{F}} \text{ and } y(B) \geq j\},$$

where

$$\mathcal{E} = \{B \subset H_0^1(\Omega) \setminus \{0\} : B \text{ is closed and } B = -B\}$$

and $\gamma : \mathcal{E} \rightarrow \mathbb{Z} \cup \{\infty\}$ is the Krasnoselski's genus function, which is defined by

$$\gamma(B) = \begin{cases} n := \min\{m \in \mathbb{N} : \text{there exists an odd map } \varphi \in C(B, \mathbb{R}^m \setminus \{0\})\}, \\ \infty, \text{ if there exists no map } \varphi \in C(B, \mathbb{R}^m \setminus \{0\}), \\ 0, \text{ if } B = \emptyset. \end{cases} \tag{2.18}$$

It is important to note that, since $\mathcal{S}_{\mathcal{F}} = -\mathcal{S}_{\mathcal{F}}$, γ_j is well defined.

Below we state some standard properties of the genus which can be found, for instance, in [18].

Lemma 2.9. *Let B and C be sets in \mathcal{E} .*

- (i) *If $x \neq 0$, then $\gamma(\{x\} \cup \{-x\}) = 1$;*
- (ii) *If there exists an odd map $\varphi \in C(B, C)$, then $\gamma(B) \leq \gamma(C)$. In particular, if $B \subset C$ then $\gamma(B) \leq \gamma(C)$.*
- (iii) *If there exists an odd homeomorphism $\varphi : B \rightarrow C$, then $\gamma(B) = \gamma(C)$. In particular, if B is homeomorphic to the unit sphere in \mathbb{R}^n , then $\gamma(B) = n$.*
- (iv) *If B is a compact set, then there exists a neighborhood $K \in \mathcal{E}$ of B such that $\gamma(B) = \gamma(K)$.*
- (v) *If $\gamma(C) < \infty$, then $\gamma(\overline{B \setminus C}) \geq \gamma(B) - \gamma(C)$.*
- (vi) *If $\gamma(A) \geq 2$, then A has infinitely many points.*

Remember that, we have denoted by $d(m)$ the sum of the dimensions of all eigenspaces V_{λ_j} associated to eigenvalues λ_j , where $1 \leq j \leq m$.

Lemma 2.10. *Suppose that $(\vartheta_1) - (\vartheta_3)$ hold and $\lambda > (\alpha^2/4)\lambda_m$. Then,*

- (i) $\gamma_{d(m)} \neq \emptyset$;
- (ii) $\gamma_1 \supset \gamma_2 \supset \dots \supset \gamma_{d(m)}$;
- (iii) *If $\varphi \in C(\mathcal{S}_{\mathcal{F}}, \mathcal{S}_{\mathcal{F}})$ and is odd, then $\varphi(\gamma_j) \subset \gamma_j$, for all $1 \leq j \leq d(m)$;*
- (iv) *If $B \in \gamma_j$ and $C \in \mathcal{E}$ with $\gamma(C) \leq s < j \leq d(m)$, then $\overline{B \setminus C} \in \gamma_{j-s}$.*

Proof. (i) Let $S_{d(m)}$ be the unit sphere of $V_1 \oplus V_2 \oplus \dots \oplus V_m$. Since $\lambda > (\alpha^2/4)\lambda_m$, it is clear that $S_{d(m)} \subset \mathcal{S}_{\mathcal{F}}$. Moreover, by Lemma 2.9(iii), we have $\gamma(S_{d(m)}) = d(m)$. Showing that $S_{d(m)} \in \gamma_{d(m)}$. (ii) It is immediate. (iii) It follows directly from Lemma 2.9(ii). (iv) It is a consequence of Lemma 2.9(v). □

Now, for each $1 \leq j \leq d(m)$, we define the following minimax levels

$$c_j = \inf_{B \in \gamma_j} \sup_{u \in B} \Psi_{\lambda,4}(u). \tag{2.19}$$

Lemma 2.11. *Suppose $(\vartheta_1) - (\vartheta_3)$ hold. Then,*

$$0 \leq c_1 \leq c_2 \leq \dots \leq c_{d(m)} < \infty.$$

Proof. First inequality follows from Lemma 2.5. On the other hand, the monotonicity of the levels c_j is a consequence of Lemma 2.10(ii). □

Next proposition is crucial to ensure the existence of multiple solutions.

Proposition 2.12. *Suppose that ϑ satisfies $(\vartheta_1) - (\vartheta_3)$ and $\lambda > (\alpha^2/4)\lambda_m$. If $c_j = \dots = c_{j+p} \equiv c$, $j + p \leq d(m)$, then $\gamma(K_c) \geq p + 1$, where $K_c := \{v \in \mathcal{S}_{\mathcal{F}} : \Psi_{\lambda,4}(v) = c \text{ and } \Psi'_{\lambda,4}(v) = 0\}$.*

Proof. Suppose that $\gamma(K_c) \leq p$. By Proposition 2.8 and Lemma 2.11, K_c is a compact set. Thus, by Lemma 2.9(iv), there exists a compact neighborhood K (in $H_0^1(\Omega)$) of K_c such that $\gamma(K) \leq p$. Defining $M := K \cap \mathcal{S}_{\mathcal{F}}$, we derive from Lemma 2.9(ii) that $\gamma(M) \leq p$. Despite the noncompleteness of $\mathcal{S}_{\mathcal{F}}$ we can still use Theorem 3.11 in [23] (see also Remark 3.12 in [23]) to ensure the existence of an odd homeomorphisms family $\eta(\cdot, t)$ of $\mathcal{S}_{\mathcal{F}}$ such that, for each $u \in \mathcal{S}_{\mathcal{F}}$, the map

$$t \mapsto \Psi_{\lambda,4}(\eta(u, t)) \text{ is non-increasing.} \tag{2.20}$$

Observe that, although $\mathcal{S}_{\mathcal{F}}$ is non-complete, from Proposition 2.7 and (2.20), for all $u \in \mathcal{S}_{\mathcal{F}}$, maps $t \mapsto \eta(u, t)$ are well defined in $t \in [0, \infty)$. Consequently, it makes sense the third claim of Theorem 3.11 in [23], namely,

$$\eta((\Psi_{\lambda,4})_{c+\varepsilon} \setminus M, 1) \subset (\Psi_{\lambda,4})_{c-\varepsilon}. \quad (2.21)$$

Let us choose $B \in \mathcal{Y}_{j+p}$ such that $\sup_B \Psi_{\lambda,4} \leq c + \varepsilon$. From Lemma 2.10(iv), $\overline{B \setminus M} \in \mathcal{Y}_j$. It follows again from Lemma 2.10(ii) that $\eta(\overline{B \setminus M}, 1) \in \mathcal{Y}_j$. Therefore, from (2.21) and the definition of c , we have

$$c \leq \sup_{\eta(\overline{B \setminus M}, 1)} \Psi_{\lambda,4} \leq c - \varepsilon,$$

that is a contradiction. Then $\gamma(K_c) \geq p + 1$. □

We are now ready to prove the following multiplicity result:

Proof of Theorem 1.2(ii):

Note that $0 \leq c_j < \infty$ are critical levels of $\Psi_{\lambda,4}$. In fact, suppose by contradiction that c_j is regular for some j . Invoking Theorem 3.11 in [23], with $\beta = c_j$, $\bar{\varepsilon} = 1$, $N = \emptyset$, there exist $\varepsilon > 0$ and a family of odd homeomorphisms $\eta(\cdot, t)$ satisfying the properties of referred theorem. Choosing $B \in \mathcal{Y}_j$ such that $\sup_B \psi < c_j + \varepsilon$ and arguing as in the proof of Proposition 2.12 we get a contradiction.

Finally, if the levels c_j , $1 \leq j \leq d(m)$, are different from each other, by Proposition 2.6(iv) the result is proved. On the other hand, if $c_j = c_{j+1} \equiv c$ for some $1 \leq j \leq d(m)$, it follows from Proposition 2.12 that $\gamma(K_c) \geq 2$. Combining last inequality with Lemma 2.9(vi) and Proposition 2.6(iv), we conclude that $(P_{\lambda,q})$ has infinitely many pairs of nontrivial solutions. □

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