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A NEW METHOD TO CALCULATE THE INCONCLUSIVE COEFFICIENTS IN THE QUANTUM STATE DISCRIMINATION

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The problem of determining the state of a quantum system is a central task in any quantum information processing. However, there are limitations imposed by quantum mechanics on the possibilities to determine the state of a quantum system. For example, nonorthogonal states cannot be discriminated perfectly. In this paper, we propose a new method to calculate the inconclusive coefficients based on the solution of a quadratic system, replacing the determination of the roots of a polynomial of degree 8, used in an algorithm to quantum state discrimination previously defined in the literature. The new method simplifies the calculation of the inconclusive coefficients and can be extended very easily to any dimension. The method was written in Matlab and successfully applied to problems with different dimensions.

Keywords: Quantum state discrimination; extended Hilbert space; semidefinite programming; quadratic systems.

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1. Introduction

The discrimination of quantum states is an essential problem in quantum information theory, and it is well known the impossibility of doing this for nonorthogonal states. However, if we allow for inconclusive results to occur, it is possible to never mistake a state by means of an optimal measurement, in terms of an appropriate Positive Operator Valued Measure (POVM), see Refs. 1 and 2. This strategy is called unambiguous state discrimination (USD), and the best procedure of this kind is that which minimizes the probability of inconclusive results. The discrimination of two nonorthogonal pure quantum states was first pointed out by Ivanovich,[3] Dieks[4] and Peres.[5] In Ref. 6, it was show that USD of N pure quantum states is possible if and only if they are linearly independent. Eldar[7] showed that the optimal measurement can be formulated as a semidefinite programming (SDP) problem, see Refs. 8 and 9. In Refs. 10–13, the USD problem is considered via the Neumark's theorem.[1]

This paper presents a new method to calculate the inconclusive coefficients in order to improve the algorithm for quantum state discrimination proposed in Ref. 13, replacing the determination of the roots of a polynomial of degree 8, by the solution of a quadratic system.[15] Furthermore, we used just one condition on the entry states: the conservation of the scalar product. The algorithm was completely rewritten in Matlab and successfully applied on problems with different dimensions (in Ref. 13, it was applied on just one problem with dimension 3).

The paper is organized as follows. First, the definition of the unambiguous state discrimination problem is given. Next, we describe the new approach. Finally, we present some computational results and ends with a brief conclusion and acknowledgments.

2. Unambiguous State Discrimination

The problem of unambiguous state discrimination (USD) for N linearly independent nonorthogonal pure quantum states can be stated as follows. We assume that a quantum system is prepared in pure quantum states from a collection of given states $\{|Q_i\rangle, i = 1, \ldots, N\}$ in an N-dimensional complex Hilbert space, i.e. given an ensemble ρ, where each state $|Q_i\rangle$ is prepared with probability μ_i,

$$\rho = \sum_{i=1}^{N} \mu_i |Q_i\rangle\langle Q_i|, \quad 0 \leq \mu_i \leq 1. \tag{1}$$

To detect the state of the system, a measurement is constructed comprising $N + 1$ measurement operators $\{\Pi_i, 0 \leq i \leq N\}$ that satisfy

$$\sum_{i=0}^{N} \Pi_i = I, \tag{2}$$

where I is the identity operator.

The measurement operators are constructed so that either the state is correctly detected, or the measurement returns an inconclusive result. Thus, each of the operators Π_i, for $i = 1, \ldots, N$, corresponds to the detection of the corresponding states $|Q_i\rangle$, and

$$\Pi_0 = I - \sum_{i=1}^{N} \Pi_i \tag{3}$$

corresponds to an inconclusive result. Because the identification must never be in error, the measurement operators must obey

$$\langle Q_i|\Pi_j|Q_i\rangle = p_i\delta_{ij}, \tag{4}$$

for $i, j = 1, \ldots, N$ and $p_i \geq 0$.

The measurement operators can be expressed in the form

$$\Pi_i = p_i\mathcal{C}_i, \quad 1 \leq i \leq N, \tag{5}$$

where $\mathcal{C}_i = |\tilde{Q}_i\rangle\langle\tilde{Q}_i|$ and the vectors $|\tilde{Q}_i\rangle$ are the states such that $\langle\tilde{Q}_i|Q_j\rangle = \delta_{ij}$, for $1 \leq i, j \leq N$. Given the matrix Ψ, whose columns are the vectors $|Q_i\rangle$, the states $|\tilde{Q}_i\rangle$ are the columns of the matrix $\tilde{\Psi}$, given by

$$\tilde{\Psi} = \Psi(\Psi^*\Psi)^{-1}. \tag{6}$$

Given an ensemble ρ, where each state $|Q_i\rangle$ is prepared with probability μ_i, the total probability of a successful detection is

$$P_D = \sum_{i=1}^{N} \mu_i\langle Q_i|\Pi_i|Q_i\rangle = \sum_{i=1}^{N} \mu_i p_i. \tag{7}$$

Therefore, the problem of optimal USD is to find measurement operators $\Pi_i = p_i\mathcal{C}_i$, or equivalently, the values p_i in order to maximize P_D subject to the constraint (2). We can express that constraint directly in terms of the values p_i as

$$I - \sum_{i=1}^{N} p_i\mathcal{C}_i \geq 0. \tag{8}$$

In Ref. 7, it was show that the optimal USD can be formulated as a semidefinite programming (SDP) problem. Generally, a SDP problem is to find $x \in R^N$ which minimizes a linear function $c^T x$, subject to a matrix inequality $F(x) = F_0 + \sum_{i=1}^{N} x_i F_i \geq 0$, where the problem data are the vector $c \in R^N$ and the $N + 1$ complex Hermitian matrices F_i.[8,9]

Expressions (7) and (8) can be put together as a SDP problem, resulting in

$$\underset{p \in R^N}{\text{minimize}}\{-\mu^T p\} \tag{9}$$

subject to the constraints

$$\begin{cases} I - \sum_{i=1}^{N} p_i \mathcal{C}_i \geq 0, \\ p_i \geq 0. \end{cases} \tag{10}$$

The solution of the problems (9) and (10) can be computed on Matlab using the CSDP package, which is based on a predictor-corrector version of the primal-dual barrier method of Helmberg *et al.*[16] It is written in C, but can be used by Matlab through yalmip toolbox.

3. The New Approach

In Ref. 13, the USD problem was considered via Neumark's theorem,[1] where the main objective was to obtain the transformation which maps N nonorthogonal pure states in a set of states that can be discriminated by usual projective measurements in an extended Hilbert space. It was used as a computational procedure that takes as input the ensemble of nonorthogonal states and outputs the best set of discriminable states.

Based on these ideas, we reformulated the algorithm replacing the calculation of the roots of a polynomial of degree 8, as it was done in Ref. 13, by the solution of a quadratic system. This system is easier to generalize to any dimension and simpler to implement in a computer code. In addition to this, it is robust when the number of variables increases.

We also start by rewriting the N entry states to be discriminated in a ladder form in the orthonormal basis $\{|i\rangle,\ i = 1, \ldots, N\}$:

$$\begin{aligned} |Q_1^{\text{lad}}\rangle &= |1\rangle, \\ |Q_2^{\text{lad}}\rangle &= c_{21}|1\rangle + c_{22}|2\rangle, \\ |Q_3^{\text{lad}}\rangle &= c_{31}|1\rangle + c_{32}|2\rangle + c_{33}|3\rangle, \\ &\vdots \\ |Q_N^{\text{lad}}\rangle &= c_{N1}|1\rangle + c_{N2}|2\rangle + c_{N3}|3\rangle + \cdots + c_{NN}|N\rangle, \end{aligned} \tag{11}$$

where the set $\{|Q_i^{\text{lad}}\rangle\}$ belongs to the Hilbert space of original size N and the coefficients c_{ij} are obtained using the preservation of the scalar product of the entry states, that is, $\langle Q_i^{\text{lad}}|Q_j^{\text{lad}}\rangle = \langle Q_i|Q_j\rangle$, for all $i, j = 1, \ldots, N$.

In order to apply the Neumark's theorem,[1] we extend the original Hilbert space to $2N - 1$ dimensions and map the original states to the final state configurations, described by Ref. 13 as follows:

$$
\begin{aligned}
|Q_{f1}\rangle &= g_1|1\rangle + g_{N+1}|N+1\rangle + \cdots + g_{2N-2}|2N-2\rangle + g_{2N-1}|2N-1\rangle \\
|Q_{f2}\rangle &= g_2|2\rangle + g_{2N}|N+1\rangle + \cdots + g_{3N-2}|2N-1\rangle \\
|Q_{f3}\rangle &= g_3|3\rangle + g_{3N-1}|N+1\rangle + \cdots + g_{4N-4}|2N-2\rangle \\
&\vdots \\
|Q_{fi}\rangle &= g_i|i\rangle + g_{i/2(2N+3-i)}|N+1\rangle + \cdots + g_{[N(i+1)+i/2(1-i)-1]}|2N+1-i\rangle \\
&\vdots \\
|Q_{fN}\rangle &= g_N|N\rangle + g_{1/2[N(N+3)-1]}|N+1\rangle.
\end{aligned} \tag{12}
$$

The state labeled by i is identified with probability g_i^2, when the measurement collapses to $|i\rangle$, for $1 \le i \le N$. For other values of i, the result is inconclusive. Therefore, the first N $g_i's$ are chosen to produce the best possible discrimination, i.e. $g_i = \sqrt{p_i}$, where the set $\{p_i\}$ is determined by the SDP problems (9) and (10).

The other coefficients $\{g_i, i = N+1 \text{ to } N/2(N+3)-1\}$ are calculated in order to preserve the normalization and the scalar products among the original states.

In Ref. 13, it was described in detail the procedure for $N = 3$, whose solution involves the calculation of the roots of a polynomial of degree 8.

In Ref. 14, the authors provided a set of equations where the optimal solution of the USD problem of N pure states must satisfy. However, despite providing explicit analytical equations to solve the problem, it is reported the difficulty of the method regarding the nonlinearity of these equations and the coupling of variables $p_1, \ldots, p_N$. The authors showed explicitly that, for $N = 3$, their method supplies a polynomial equation of degree 6. One point that has not been studied is about the relation between an arbitrary N and the degree of the associated polynomial equation.

Now, we will present a new strategy for the determination of the inconclusive coefficients for a generic N,[15] resulting in a more robust procedure that eliminates the need to solve a polynomial equation of degree 6 or 8, previously reported in the literature.

First, we propose a new nomenclature for the final discriminable configuration, which will facilitate the understanding of the resulting quadratic system. Rewriting the system (12), we get

$$
\begin{aligned}
|Q_{f1}\rangle &= g_{11}|1\rangle + g_{1,N+1}|N+1\rangle + \cdots + g_{1,2N-2}|2N-2\rangle + g_{1,2N-1}|2N-1\rangle \\
|Q_{f2}\rangle &= g_{22}|2\rangle + g_{2,N+1}|N+1\rangle + \cdots + g_{2,2N-2}|2N-2\rangle + g_{2,2N-1}|2N-1\rangle \\
|Q_{f3}\rangle &= g_{33}|3\rangle + g_{3,N+1}|N+1\rangle + \cdots + g_{3,2N-2}|2N-2\rangle \\
&\vdots \\
|Q_{fi}\rangle &= g_{ii}|i\rangle + g_{i,N+1}|N+1\rangle + \cdots + g_{i,2N+1-i}|2N+1-i\rangle \\
&\vdots \\
|Q_{fN}\rangle &= g_{NN}|N\rangle + g_{N,N+1}|N+1\rangle,
\end{aligned} \tag{13}
$$

which in the matrix form can be given by

$$Q_f = \big(|Q_{f1}\rangle \quad |Q_{f2}\rangle \quad \cdots \quad |Q_{fN}\rangle\big) \tag{14}$$

$$= \begin{pmatrix} g_{11} & 0 & \cdots & 0 & \cdots & 0 \\ 0 & g_{22} & \cdots & 0 & \cdots & 0 \\ 0 & 0 & \ddots & 0 & \cdots & 0 \\ 0 & 0 & \ldots & g_{ii} & \ldots & 0 \\ \vdots & \vdots & \ldots & \vdots & \ddots & 0 \\ 0 & 0 & \ldots & 0 & \ldots & g_{NN} \\ g_{1,N+1} & g_{2,N+1} & \cdots & g_{i,N+1} & \cdots & g_{N,N+1} \\ g_{1,N+2} & g_{2,N+2} & \cdots & g_{i,N+2} & \cdots & 0 \\ g_{1,N+3} & g_{2,N+3} & \cdots & g_{i,2N+1-i} & \cdots & 0 \\ \vdots & \vdots & \ldots & 0 & \ldots & 0 \\ \vdots & \vdots & \ldots & 0 & \ldots & 0 \\ g_{1,2N-1} & g_{2,2N-1} & \cdots & 0 & \ldots & 0 \end{pmatrix}, \tag{15}$$

of dimension $2N - 1 \times N$.

Recall that the elements g_{ii} of Q_f, for $i = 1, \ldots, N$, referring to the components related to the orthonormal basis are obtained by solving a SDP problem. Thus, the only unknown elements in this matrix are the coefficients g_{ij} associated with the extended Hilbert space, for $i = 1, \ldots, N$ and $j = N + 1, \ldots, 2N - 1$. We remark that this is the minimum possible extension, since the minimum dimensionality of the ancilla space is $N - 1$, as it was proved in Ref. 17.

To solve this problem, we propose the use of basic concepts of linear algebra, using matrices in block form. Therefore, rewriting in the block form the matrices Q_f and $\tilde{Q}^{\text{lad}}$ of dimension $2N - 1 \times N$ by

$$\tilde{Q}^{\text{lad}} = \begin{pmatrix} Q^{\text{lad}} \\ \cdots \\ \tilde{\text{O}} \end{pmatrix} = \begin{pmatrix} 1 & c_{21} & c_{31} & \ldots & c_{N1} \\ 0 & c_{22} & c_{32} & \ldots & c_{N2} \\ 0 & 0 & c_{33} & \ldots & c_{N3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 0 \end{pmatrix} = \begin{pmatrix} 1 & \vdots & s^t \\ \cdots & \cdots & \cdots \\ 0 & \vdots & S \\ \cdots & \cdots & \cdots \\ 0 & \vdots & O \end{pmatrix}, \tag{16}$$

where

$$s = \begin{pmatrix} c_{21} \\ c_{31} \\ \vdots \\ c_{N1} \end{pmatrix}$$

is a vector of dimension $N-1$, 0 is a null vector of dimension

$$N-1, \quad S = \begin{pmatrix} c_{22} & c_{32} & \dots & c_{N2} \\ 0 & c_{33} & \dots & c_{N3} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & c_{NN} \end{pmatrix}$$

is a matrix of dimension $N-1 \times N-1$ and O is a null matrix of dimension $N-1 \times N-1$.

The matrix Q_f in the block form is given by

$$Q_f = \begin{pmatrix} g_{11} & \vdots & 0^t \\ \cdots & \cdots & \cdots \\ 0 & \vdots & \tilde{D}^t \\ \cdots & \cdots & \cdots \\ b & \vdots & B^t \end{pmatrix}, \tag{17}$$

where 0 is a null vector of dimension

$$N-1, \quad \tilde{D} = \begin{pmatrix} g_{22} & 0 & \dots & 0 \\ 0 & g_{33} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & g_{NN} \end{pmatrix},$$

is a matrix of dimension

$$N-1 \times N-1, \quad B^t = \begin{pmatrix} g_{2,N+1} & g_{3,N+1} & \cdots & g_{N-1,N+1} & g_{N,N+1} \\ g_{2,N+2} & g_{3,N+2} & \cdots & g_{N-1,N+2} & 0 \\ \vdots & \vdots & \dots & 0 & 0 \\ g_{2,2N-1} & 0 & \dots & 0 & 0 \end{pmatrix}$$

is also a matrix of dimension

$$N-1 \times N-1 \quad \text{and} \quad b = \begin{pmatrix} g_{1,N+1} \\ g_{1,N+2} \\ \vdots \\ g_{1,2N-1} \end{pmatrix}$$

is a vector of dimension $N-1$.

By the conservation of the scalar product between the column vectors of the matrices Q_f and $\tilde{Q}^{\text{lad}}$, i.e.

$$Q_f^t Q_f = \tilde{Q}^{\text{lad}^t} \tilde{Q}^{\text{lad}}, \tag{18}$$

we obtain

$$\begin{pmatrix} g_{11} & \vdots & 0^t & \vdots & b^t \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \vdots & \tilde{D} & \vdots & B \end{pmatrix} \begin{pmatrix} g_{11} & \vdots & 0^t \\ \cdots & \cdots & \cdots \\ 0 & \vdots & \tilde{D}^t \\ \cdots & \cdots & \cdots \\ b & \vdots & B^t \end{pmatrix}$$

$$= \begin{pmatrix} 1 & \vdots & 0^t & \vdots & 0^t \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ s & \vdots & S^t & \vdots & O^t \end{pmatrix} \begin{pmatrix} 1 & \vdots & s^t \\ \cdots & \cdots & \cdots \\ 0 & \vdots & S \\ \cdots & \cdots & \cdots \\ 0 & \vdots & O \end{pmatrix}, \tag{19}$$

which implies that

$$\begin{pmatrix} g_{11}^2 + b^t b & \vdots & b^t B^t \\ \cdots & \cdots & \cdots \\ Bb & \vdots & \tilde{D}\tilde{D}^t + BB^t \end{pmatrix} = \begin{pmatrix} 1 & \vdots & s^t \\ \cdots & \cdots & \cdots \\ s & \vdots & ss^t + S^t S \end{pmatrix}. \tag{20}$$

Considering the matrix equalities, we get

$$\begin{cases} g_{11}^2 + b^t b = 1, \\ b^t B^t = s^t, \\ Bb = s, \\ \tilde{D}\tilde{D}^t + BB^t = S^t S + ss^t. \end{cases} \tag{21}$$

That is, our problem consists in solving the following quadratic system

$$\begin{cases} BB^t = S^t S + ss^t - \tilde{D}\tilde{D}^t, \\ Bb = s, \\ g_{11}^2 + b^t b = 1, \end{cases} \tag{22}$$

whose solution will provide the elements g_{ij} associated with the basis of the extended Hilbert space, for $i = 1, \ldots, N$ and $j = N+1, \ldots, 2N-1$.

4. Computational Results

Before presenting some computational results for many values of N, we will see an example of the application of all the procedure for $N = 3$, where we provide the analytical solution of the problem.

Consider the ensemble $\{(|Q_1\rangle, \mu_1); (|Q_2\rangle, \mu_2); (|Q_3\rangle, \mu_3)\}$, with $\sum_{i=1}^{3} \mu_i = 1$. In order to obtain the final discriminable configuration, we have the following steps:

- First, we start by rewriting the entry states to be discriminated in a ladder form in the orthonormal basis $\{|i\rangle, i = 1, 2, 3\}$:

$$\begin{cases} |Q_1^{\mathrm{lad}}\rangle = |1\rangle, \\ |Q_2^{\mathrm{lad}}\rangle = c_{21}|1\rangle + c_{22}|2\rangle, \\ |Q_3^{\mathrm{lad}}\rangle = c_{31}|1\rangle + c_{32}|2\rangle + c_{33}|3\rangle. \end{cases} \tag{23}$$

- Solving the problems (9) and (10), we obtain

$$p = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix}, \tag{24}$$

  where $g_{ii} = \sqrt{p_i}$, for $i = 1, 2, 3$.
- In order to calculate the coefficients g_{ij}, for $i = 1, 2, 3$ and $j = 4, 5$, we have to solve the system (22), given by

$$\begin{pmatrix} \sqrt{p_1} & 0 & 0 & g_{14} & g_{15} \\ 0 & \sqrt{p_2} & 0 & g_{24} & g_{25} \\ 0 & 0 & \sqrt{p_3} & g_{34} & 0 \end{pmatrix} \begin{pmatrix} \sqrt{p_1} & 0 & 0 \\ 0 & \sqrt{p_2} & 0 \\ 0 & 0 & \sqrt{p_3} \\ g_{14} & g_{24} & g_{34} \\ g_{15} & g_{25} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ c_{21} & c_{22} & 0 & 0 & 0 \\ c_{31} & c_{32} & c_{33} & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & c_{21} & c_{31} \\ 0 & c_{22} & c_{32} \\ 0 & 0 & c_{33} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

  which implies that (recall that $c_{21}^2 + c_{22}^2 = 1$ and $c_{31}^2 + c_{32}^2 + c_{33}^2 = 1$)

$$p_1 + g_{14}^2 + g_{15}^2 = 1, \tag{25}$$

$$g_{14}g_{24} + g_{15}g_{25} = c_{21}, \tag{26}$$

$$g_{14}g_{34} = c_{31}, \tag{27}$$

$$p_2 + g_{24}^2 + g_{25}^2 = c_{21}^2 + c_{22}^2 \Rightarrow p_2 + g_{24}^2 + g_{25}^2 = 1, \tag{28}$$

$$g_{24}g_{34} = c_{21}c_{31} + c_{22}c_{32}, \tag{29}$$

$$p_3 + g_{34}^2 = c_{31}^2 + c_{32}^2 + c_{33}^2 \Rightarrow p_3 + g_{34}^2 = 1. \tag{30}$$

From Eq. (30) we obtain $g_{34} = \sqrt{1 - p_3}$, which implies that $g_{24} = (c_{21}c_{31} + c_{22}c_{32})/\sqrt{1 - p_3}$ by Eq. (29). Using this value for g_{24} in (28), we get $g_{25} = \sqrt{1 - p_2 - (c_{21}c_{31} + c_{22}c_{32})^2/(1 - p_3)}$. Now, using the value for g_{34} in (27), we obtain $g_{14} = c_{31}/\sqrt{1 - p_3}$. Finally, using the values for g_{14}, g_{24}, g_{25} in (26), we obtain $g_{15} = (c_{21} - c_{31}(c_{21}c_{31} + c_{22}c_{32})/(1 - p_3))/\sqrt{1 - p_2 - (c_{21}c_{31} + c_{22}c_{32})^2(1 - p_3)}$. That is, the final discriminable configuration, in terms of $c_{21}, c_{22}, c_{31}, c_{32}, c_{33}$ and p_1, p_2, p_3, is given by

$$\begin{cases} |Q_{f1}\rangle = \sqrt{p_1}|1\rangle + \dfrac{c_{31}}{\sqrt{1 - p_3}}|4\rangle + \dfrac{c_{21} - \frac{c_{31}(c_{21}c_{31} + c_{22}c_{32})}{1 - p_3}}{\sqrt{1 - p_2 - \frac{(c_{21}c_{31} + c_{22}c_{32})^2}{1 - p_3}}}|5\rangle, \\ |Q_{f2}\rangle = \sqrt{p_2}|2\rangle + \dfrac{c_{21}c_{31} + c_{22}c_{32}}{\sqrt{1 - p_3}}|4\rangle + \sqrt{1 - p_2 - \dfrac{(c_{21}c_{31} + c_{22}c_{32})^2}{1 - p_3}}|5\rangle, \\ |Q_{f3}\rangle = \sqrt{p_3}|3\rangle + \sqrt{1 - p_3}|4\rangle. \end{cases} \tag{31}$$

In Table 1, we present the medium time, considering 10 random instances for different values of N, necessary to obtain the final discriminable configuration, using the procedure described in this paper. The code was written in Matlab 7.0.1 and all the experiments were carried out on an Intel Core 2, 1.66 GHz and 1 GB RAM, running Windows XP.

From that table, we can see that our procedure is able to obtain the final discriminable configuration for different values of N, in a very reasonable time. In

Table 1. Medium time for different values of N.

N	Time
2	1.4344s
3	1.7156s
4	1.7706s
5	2.1174s
6	2.0282s
7	2.2936s
8	2.9421s
9	2.4860s
10	2.0514s
20	2.6954s
30	4.3342s
40	6.1859s
50	9.0860s
60	14.8456s
70	20.8436s
80	32.8407s
90	44.4249s
100	1min 11.025s

addition to this, the new method proposed to calculate the inconclusive coefficients is easier to implement and more robust than the one proposed in Ref. 13.

5. Conclusion

We proposed a new method to calculate the inconclusive coefficients of the final discriminable configuration calculated by the algorithm described by Ref. 13, replacing the determination of the roots of a polynomial of degree 8 by the solution of a quadratic system. The new method simplified the calculation of the inconclusive coefficients and could be extended very easily to any dimension. The method was written in Matlab and successfully applied to problems with dimensions varying from $N = 2$ to $N = 100$.

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